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INVESTIGATION OF DEFORMATION AND FRACTURE MECHANICS OF A VISCOELASTIC BEAM WITH A RECTANGULAR CROSS-SECTION

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Abstract: As the use of polymer and composite materials expands across aviation, highway infrastructure, and civil engineering, understanding the complex viscoelastic behaviors that defy classical elasticity theory becomes essential. This research investigates the deformation and failure mechanisms of rectangular viscoelastic beams under time-dependent loading. The paper integrates the Euler-Bernoulli bending theory with the Standard Linear Solid (SLS) model, while applying the Kachanov-Rabotnov damage model to calculate micro-defect accumulation and time-to-rupture. Finally, the study presents practical case studies on the flexural behavior of glass fiber-reinforced polymer (GFRP) bars and peridynamic fracture analysis.

Keywords: *Viscoelasticity, creep, rectangular cross-section, Kachanov-Rabotnov model, bending deformation, damage mechanics, composite materials*.

Introduction

Predicting how long a load-bearing structure will survive requires a deep understanding of time-dependent material deformation. Rectangular cross-section beams form the backbone of most civil and infrastructure frameworks, from highway bridge girders to high-rise structural supports. When these components face sustained loading, they do not just sit passively; they deform continuously over time. This phenomenon is known as creep.

Engineers have historically relied on Hooke's Law and standard elasticity theory to design these structures. However, timber, modern polymer composites, and heavily loaded concrete exhibit clear viscoelastic behavior. They suffer from a continuous, hidden accumulation of micro-damage under load. If left unchecked or improperly calculated, this slow degradation abruptly transitions into creep rupture — a total structural failure. The core objective of this study is to move beyond static elastic models and provide a rigorous mathematical description of how rectangular beams bend and eventually fail under prolonged stress.

Viscoelastic Behavior and Flexural Mechanics

To accurately predict long-term structural response, we must look at how materials combine immediate elastic resistance with delayed fluid-like yielding.

The Standard Linear Solid Model

Instead of simple springs or dashpots, engineering applications generally rely on the Standard Linear Solid (SLS) model (or Maxwell-Wiechert model) to capture real-world material behavior. For linear viscoelastic materials, the strain ϵ at any given time t is influenced by the entire history of applied stress σ. We express this using the Boltzmann superposition integral:

$$\epsilon(t) = \int_0^t J(t-\tau)\frac{\partial\sigma(\tau)}{\partial\tau}\,d\tau \tag{1}$$

Here, $J(t)$ acts as the material's creep compliance function. In the SLS framework, compliance takes the following exponential form:

$$J(t) = J_0 + J_1\left(1 - \exp\left(-\frac{t}{\tau_c}\right)\right) \tag{2}$$

In this relation, J_0 dictates the immediate elastic snap, J_1 represents the slow viscoelastic yield over time, and τ_c defines the material's inherent relaxation time.

Bending in Rectangular Beams

Let us consider a structural beam with a rectangular cross-section of width t and height h. Its area A and moment of inertia I around the neutral axis are standard geometric properties:

$$A = bh, \qquad I = \frac{bh^3}{12} \tag{3}$$

Following the classic Euler-Bernoulli assumption, cross-sections remain plane and perpendicular to the neutral axis during bending. This means the longitudinal strain varies linearly across the beam's height:

$$\epsilon(x, y, t) = -yk(x, t) \tag{4}$$

where $k(x, t)$ is the evolving curvature. A changing bending moment $M(x, t)$ forces internal stresses to balance out across the section:

$$M(x, t) = \int_{-h/2}^{h/2} \sigma(x, y, t) bydy \tag{5}$$

If we substitute the viscoelastic strain equation into this equilibrium condition, we arrive at the governing equation for the beam's curvature over time:

$$k(x, t) = \frac{1}{I} \int_0^t J(t - \tau) \frac{\partial M(x, \tau)}{\partial \tau} \, d\tau \tag{6}$$

This mathematical result highlights a major engineering challenge: even if a bridge girder carries a perfectly constant dead load M_0, its curvature k will persistently increase because the compliance function $J(t)$ grows over time.

Progressive Damage and Rupture

A structure does not just sag indefinitely; eventually, the internal polymer chains or physical fibers break.

Modeling Steady-State Creep

Creep generally unfolds in three phases: an initial rapid phase, a long steady-state phase, and a final accelerated phase leading to collapse. Structural design mostly concerns the long secondary phase, which is best captured by the Findley power-law:

$$\epsilon(t) = \epsilon_0 + mt^n \tag{7}$$

where ϵ_0 is the instant strain, and m and n are empirical constants derived from the specific material and environmental temperature.

The Kachanov-Rabotnov Approach to Failure

While Findley's law handles the steady deformation, it cannot predict the exact moment of failure. To do that, we introduce Continuum Damage Mechanics. The Kachanov-Rabotnov model defines a damage parameter, ω. A virgin material has an ω of 0, while a completely fractured element has an ω of 1.

As micro-cracks form, the cross-sectional area that actually bears the load shrinks. This forces the remaining intact material to handle a higher "effective stress":

$$\tilde{\sigma} = \frac{\sigma}{1 - \omega} \tag{8}$$

Damage does not grow linearly. The rate at which the beam deteriorates accelerates according to this differential equation:

$$\frac{d\omega}{dt} = C\left(\frac{\sigma}{1 - \omega}\right)^q \tag{9}$$

where C and q are material constants. By integrating this equation from a pristine state $(\omega = 0,\ t = 0)$ to total failure $(\omega = 1,\ t = t_R)$, we can pinpoint the exact rupture time t_R:

$$t_R = \frac{1}{C(q + 1)\sigma^q} \tag{10}$$

This specific equation is a warning sign for structural design. Because stress σ is raised to the power of q, even a marginal overload at the beam's outer edges $y = \pm h/2$ will drastically slash the component's lifespan.

Practical Implications in Engineering

Translating these equations into real-world civil engineering reveals why classical elastic designs often fail prematurely.

Deflection in GFRP Reinforcements

Glass Fiber-Reinforced Polymer (GFRP) bars are increasingly replacing traditional steel rebar, especially in environments where corrosion is a major risk. However, when a rectangular FRP element faces sustained bending, the polymer matrix creeps.

A recent study evaluating hybrid composite bars under a 50-year sustained load utilizing Findley parameters exposed severe long-term deviations from initial elastic estimates. While the short-term maximum deflection might register at 12.5 mm, the predicted deflection after 50 years climbs to 15.8 mm—a nearly 26% increase. Concurrently, stress relaxation drops internal resistance by over 14%, pushing the damage index ω dangerously close to critical thresholds. Designing these elements strictly elastically guarantees eventual serviceability failure.

Shifting to Peridynamic Fracture Analysis

Traditional fracture mechanics breaks down mathematically right at the tip of a crack, producing infinite stress values that are impossible to resolve. To bypass this, researchers have recently started using Peridynamics for viscoelastic materials.

Instead of traditional derivatives, peridynamics uses integration. The movement of a specific point x in the beam is calculated by summing up the physical forces exerted by all nearby points within a specific radius:

$$\rho u(x,t) = \int_{H_x} f(u(x',t) - u(x,t),\, x' - x) dV_{x'} + b(x,t) \tag{11}$$

When the SLS viscoelastic model is fed into this peridynamic framework, the simulation naturally forms micro-cracks. It can predict exactly where a beam will begin to fracture internally without an engineer having to guess where the crack will start.

Conclusion

Relying purely on Hooke's Law to design rectangular beams—especially those made of polymers, timber, or composites—is mathematically insufficient and practically dangerous. Time is a distinct mechanical variable that dictates structural integrity.

By combining the Euler-Bernoulli bending theory with the Boltzmann superposition principle, we can accurately track how a beam sags over decades. More importantly, applying the Kachanov-Rabotnov damage equations allows us to calculate the precise threshold where steady creep turns into catastrophic rupture. As infrastructure modernizes and new composite materials are deployed, integrating these non-linear, time-dependent models into standard engineering practice is non-negotiable for ensuring public safety.

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