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PROBLEMS OF SPHERICAL MOTION KINEMATICS OF A RIGID BODY

N.A.NASIBOV, V.I. BAKHSHALI, A.N.NOVRUZOV
Azerbaijan Technical University,
v.bakhshali@aztu.edu.az

Abstract.The article proposes theorems that consider the properties of accelerations of points of a rigid body in its spherical motion. The existence of planes invariably connected with a rigid body performing spherical motion, the normal and tangential accelerations of points, which are equal to the axial and rotational accelerations of these points, respectively, are shown. Special attention is paid to the question of the direction of the vector of normal acceleration of the points of this body. In some cases, comparative features of rotational and axial acceleration on the one hand, and tangential and normal accelerations on the other, are shown.
Keywords: rigid body, spherical motion, kinematics, theorem, normal acceleration, tangential acceleration.

Introduction

The study of the peculiarities of spherical motion (rotation around a fixed point) of a rigid body (RB) is not only theoretical, but also of practical interest (gyroscopes, rockets, cosmic bodies, drill bit cutters, etc.) and is one of the most important tasks of theoretical mechanics.

An example of a spherical body motion would be a gyroscope motion in a cardan suspension or a top motion of which pointed end rests on a stand and remains stationary.

When studying the kinematics of the spherical motion of a body, it is necessary to establish the main characteristics of this motion, that is, the equations of motion, angular velocity, and angular acceleration of the body, and derive formulas for calculating the velocities and acceleration points of the body. Below we consider the problem for investigating the kinematics of spherical motion with respect to the instantaneous axis method [1-7].

The further development of the theory of this motion is an actual problem of Mechanics. In this regard, this article presents theorems and their proofs devoted to the kinematics of rigid body points at spherical motion. The result of the research can be useful for the design and calculation of machine parts subjected to spherical motion.

1. Analysis to substantiate the kinematics of spherical motion of rigid body

Theorem 1. The vector of normal acceleration of any point of a solid body making spherical motion is located in a plane passing through this point and the instantaneous axis of rotation of this body, is directed to the concave side of the sphere passing through this point with the center at the stationary point of the body.

Proof. Let us consider the motion of an arbitrary point M of RB making a spherical motion around a fixed center O (Fig. 1). Let OP be the instantaneous axis of rotation of this body. In this case, velocity $\bar{v}_M$ of the M point of the body, hence the tangential acceleration $\bar{w}_M^t$ of this point is perpendicular to the plane of the MPA, which is the normal plane. Then normal acceleration $\overline{w}_M^n$ of point M will be located in the same plane.

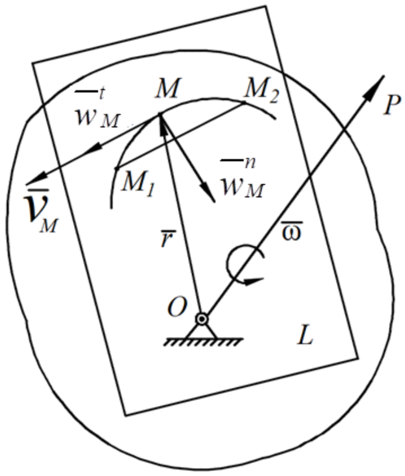

Fig. 1. The motion of an arbitrary point of RB making a spherical motion around a fixed center

Let us now consider the proof of the second part of the theorem. First, we show that the center of curvature of any line on a spherical surface is on the concave side of that surface. In this case, we mean a spherical surface centered at point O and radius OM. Let us take the points M on the trajectory, and the points M_1 and M_2 located near it. Let us draw the plane *L* through these points (this plane does not generally pass through the center O). Since the plane L passes through three points of the spherical surface, it intersects the last one. This remains true if the density P is transformed into a contiguous plane, i.e., the points M_1 and M_2 are infinitely approximated to the point M. In any case, the chord M_1M_2 will remain inside the spherical surface. Then the principal normal of the trajectory at its point M, and consequently, the normal acceleration of the point will be directed to the inner, i.e. to the concave side of the given spherical surface. The theorem has been proved.

Theorem 2. In the spherical motion of a solid body at each moment of time there is a plane invariably connected with this body and passing through the fixed center, the axial and rotational acceleration of each point, which is equal to the normal and tangential accelerations of this point, respectively.

Proof. Let the rigid body make a spherical motion, and rotate around some fixed center O. Let us assume that $\overline{\omega}$ and $\overline{\varepsilon}$ are respectively vectors of instantaneous angular velocity and instantaneous angular acceleration of this body (Fig. 2). Let us consider the motion of an arbitrary point M of the body.

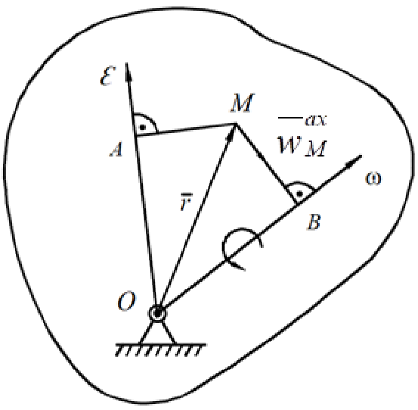

Fig. 2. Demonstration of the instantaneous angular velocity and instantaneous angular acceleration of RB

As you know, the tangential acceleration of a point is directed along the line of action of its velocity vector. A is the velocity vector of the point M of the body

$$w_M = \bar{\omega} \times \bar{r} \quad (1)$$

Where $\bar{r}$ - is the radius vector of the point M of the body.
The rotational acceleration of this point

$$w_M^{rt} = \bar{\varepsilon} \times \bar{r} \quad (2)$$

From the foregoing, it follows that the tangential acceleration of point M is perpendicular to the plane OMB, and the rotational acceleration is perpendicular to the plane OMA. Therefore, in order for the tangential and rotational acceleration of point M to coincide in direction, the planes of the triangles OMA and OMB must coincide. This is possible only when the point M is located in the plane of vectors $\bar{\omega}$ and $\bar{\varepsilon}$.
Axial acceleration of the M point of the body

$$\bar{w}_M^{ax} = \bar{w} \times \bar{v}_M \quad (3)$$

According to this expression $\bar{w}_M^{ax}$, is directed in a straight line MB, perpendicular to the vector $\bar{\omega}$. Thus, the rotational acceleration of point M located in the plane of vectors $\bar{\omega}$ and $\bar{\varepsilon}$, is directed along the velocity vector of this point, and the axial acceleration $\bar{w}_M^{ax}$ is perpendicular to the rotational acceleration (since the vector $\bar{w}_M^{ax}$ located in the vector plane $\bar{\omega}$ and $\bar{\varepsilon}$). It follows that $\bar{w}_M^{rt}$ at the same time, it is tangential, and $\bar{w}_M^{ax}$ since the point M occupies an arbitrary position in the plane of vectors $\bar{\omega}$ and $\bar{\varepsilon}$, the resulting conclusion can be applied to all points of this plane. The theorem has been proved.

Theorem 3. In the spherical motion of a solid body at each moment of time there is a line invariably connected with this body and passing through a fixed center, the modulus of acceleration of each point of which can be determined in the same way as in the rotation of the body around a fixed axis, i.e., according to the formula

$$w = r\sqrt{\varepsilon^2 + \omega^4}.$$

Proof. Let us suppose that the RB revolves around some fixed center O, i.e., that it is in spherical motion (Fig. 3).

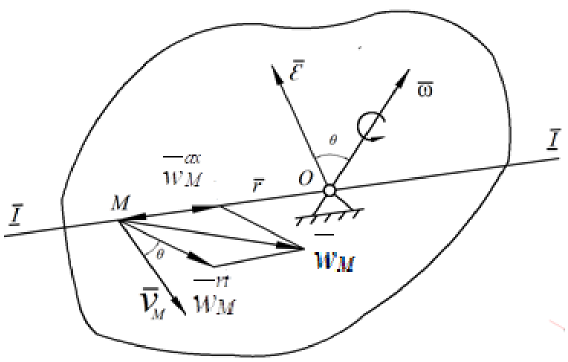

Fig. 3. Demonstration of the acceleration of the arbitrary point of RB

Let us draw a straight line I-I passing through the fixed center O perpendicular to the plane of vectors $\bar{\omega}$ and $\bar{\varepsilon}$. Let us consider an arbitrary point M of a body that coincides with this line at the given time. Let $\overline{OM} = \bar{r}$. Then, according to expression (2), the rotational acceleration of point M of the body will be directed perpendicular to the line I-I. Because, in this case, $\bar{\omega}$ and $\bar{\varepsilon}$ are perpendicular, the module of this acceleration will be $w_M^{rt} = \varepsilon r$. Axial acceleration $\overline{w}_M^{ax}$ of point M is directed on lines I-I to the fixed center O. As a result, between the acceleration vectors $\overline{w}_M^{rt}$ and $\overline{w}_M^{ax}$, the right angle is formed. Then the total acceleration of the point M of the body is

$$w_M = \sqrt{\left(w_M^{rt}\right)^2 + \left(w_M^{ax}\right)^2} \quad (4)$$

The axis acceleration modulus of the point M of the body will be $\overline{w}_M^{ax} = \omega^2 r$. Substituting the given values $\overline{w}_M^{rt}$ and $\overline{w}_M^{ax}$ in formula (4), we get:

$$w_M = r\sqrt{\varepsilon^2 + \omega^4} \quad (5)$$

Since point M can occupy any position on line I-I, the expression (5) is valid for all points of the body that coincide with this line at the time under consideration. The theorem has been proved.

It should be noted that although the expressions of the moduli of rotational and axial acceleration, in this case, coincide with the expressions of the tangential and normal acceleration of the points RB rotating around a fixed axis, the vectors $\overline{w}_M^{rt}$ and $\overline{w}_M^{ax}$, in the case of any non-zero value (except 180^0) of the angle θ between the vectors $\overline{\omega}$ and $\overline{\varepsilon}$ are not tangential and normal accelerations, respectively.

2. Practical applications of theorem 2 which is proved above

Using theorem 2, it is possible to find the value of the radius of curvature of the trajectory of any point of the body at the moment when this point coincides with the plane of vectors $\overline{\omega}$ and $\overline{\varepsilon}$. Let $MB = h$. In accordance with (3), axial acceleration $w_M^{ax} = \omega v_M = \omega^2 h$, and normal acceleration $w_M^n = \frac{v_M^2}{\rho} = \frac{\omega^2 h^2}{\rho}$, where ρ is the radius of curvature of the trajectory of the point M of the body at the specified moment. According to the theorem, at this point $w_M^{ax} = w_M^n$. Then, equating the right parts of the given equations, we determine $\rho = h$.

3. Results and discussions

Considered properties of the accelerations of the points of a rigid body in spherical motion. The existence of planes invariably connected with a rigid body performing spherical motion, the normal and tangential accelerations of points, which are equal to the axial and rotational accelerations of these points, respectively, are shown. Special attention is paid to the question of the direction of the vector of normal acceleration of the points of this body.

The results of the study can be useful in the calculation and design of elements of machines and equipment that perform spherical movements.

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